Question 1 (Section 5 Question 15 a, b, c)

If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$ compute:

$$(a)\ \mathbb{P}(X > 5),\ \ (b)\ \mathbb{P}(4 < X < 16),\ \ (c)\ \mathbb{P}(X < 8)$$

Solution

For these questions we use the trick that $Z = \frac{X-\mu}{\sigma}$ and use the standard normal table (or R!) I include R codes for the final answer, but we go over the table during tutorial. You'd need to use the table for tests.

(a)

$$\mathbb{P}(X > 5) = \mathbb{P}\left(\frac{X-\mu}{\sigma} > \frac{5-10}{\sqrt{36}}\right) = \mathbb{P}\left(Z > \frac{-5}{6}\right) = \mathbb{P}\left(Z < \frac{5}{6}\right)$$
$$\approx \mathbb{P}\left(Z < 0.83\right)$$

```
> pnorm(0.8333333)
[1] 0.7976716
```

(b)

$$\begin{aligned}\mathbb{P}(4 < X < 16) &= \mathbb{P}(X < 16) - \mathbb{P}(X < 4)\\ &= \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{16-10}{\sqrt{36}}\right) - \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{4-10}{\sqrt{36}}\right)\\ &= \mathbb{P}(Z < 1) - \mathbb{P}(Z < -1) = 2 * \mathbb{P}(Z < 1) - 1\end{aligned}$$

```
> 2*(pnorm(1)) - 1
[1] 0.6826895
```

(c)

$$\mathbb{P}(X < 8) = \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{8-10}{\sqrt{36}}\right) = \mathbb{P}\left(Z < \frac{-1}{3}\right) = 1 - \mathbb{P}\left(Z < \frac{1}{3}\right)$$

```
1 - pnorm(1/3)
[1] 0.3694414
```

Question 2 (Section 5 Question 18)

Suppose that X is a normal random variable with mean 5. If $\mathbb{P}(X > 9) = 0.2$, approximately what is $Var(X)$?

Solution

$$\begin{aligned}\mathbb{P}(X > 9) &= 0.2\\ 1 - \mathbb{P}(X > 9) &= 1 - 0.2\\ \mathbb{P}(X < 9) &= 0.8\\ \mathbb{P}\left(Z < \frac{9-5}{\sigma}\right) &= 0.8\end{aligned}$$

From the table, to get a $0.7995 \approx 0.8$ probability we need a Z score of 0.84. Hence,

$$\frac{9-5}{\sigma} = 0.84 \quad \Rightarrow \quad \sigma^2 = \left(\frac{4}{0.84}\right)^2 = 22.67574$$

Hence $Var(X) = \sigma^2 = 22.67574$.

Remark 1: $Var(X) = \sigma^2$ occurs for a normal distribution; this does not apply to other ones.

Remark 2: the answer is slightly different from the answer key, but it does depend on the decimals you keep. If you were to try to get the precise answer through code, it'd also be slightly different.

```
> ((9-5)/qnorm(0.8))^{2}
[1] 22.58846
```

Question 3 (Section 5 Question 21)

Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$.

(a) What percentage of 25-year-old men are taller than 6 feet, 2 inches?

(b) What percentage of men in the 6-footer club are taller than 6 feet, 5 inches?

Solution

1. 1 feet represents 12 inches, so 6 feet and 2 inches is equivalent to $6 * 12 + 2 = 74$ inches.

$$\mathbb{P}(X > 74) = 1 - \mathbb{P}(X < 74) = 1 - \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{74-71}{\sqrt{6.25}}\right)$$

$$= 1 - \mathbb{P}(Z < 1.2) = 1 - 0.8849303 = 0.1150697$$

```
> 1 - pnorm((74 - 71)/sqrt(6.25))
[1] 0.1150697
```

2. We are looking for $6 * 12 + 5 = 77$ inches. Furthermore, we're focusing on people who are at least $6 * 12 = 72$ inches. Note that:

$$\mathbb{P}(X > 77) = 1 - \mathbb{P}(X < 77) = 1 - \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{77-71}{\sqrt{6.25}}\right)$$

$$= 1 - \mathbb{P}(Z < 2.4) = 1 - 0.9918025 = 0.008197536$$

Similarly,

$$\mathbb{P}(X > 72) = 1 - \mathbb{P}(X < 72) = 1 - \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{72-71}{\sqrt{6.25}}\right)$$

$$= 1 - \mathbb{P}(Z < 0.4) = 1 - 0.6554217 = 0.3445783$$

Hence,

$$\mathbb{P}(X > 77 | X > 72) = \frac{\mathbb{P}(X > 77)}{\mathbb{P}(X > 72)} = \frac{0.008197536}{0.3445783} = 0.02379006$$

```
> num = 1 - pnorm((77 - 71)/sqrt(6.25))
> denom = 1 - pnorm((72 - 71)/sqrt(6.25))
> num/denom
[1] 0.02379006
```

Question 4 (Section 5 Question 25)

Two types of coins are produced at a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas if it lands on heads fewer than 525 times, then we shall conclude that it is a fair coin.

(a) If the coin is actually fair, what is the probability that we shall reach a false conclusion?

(b) What would it be if the coin were biased?

Solution Part 1

Here we will use the binomial approximation to a normal distribution, i.e., using the property that if S_n denotes the number of successes then:

$$\frac{S_n - np}{\sqrt{np(1-p)}} \sim N(0,1)$$

Let S_n denote the number of heads we see. Note that this satisfies assumption:
$n(p)(1-p) = 1000(0.5)(0.5) = 250 \geq 10, \ \ 1000(0.45)(0.55) = 247.5 \geq 10.$

(a) Reaching a false conclusion means that we say this coin is biased, which means we had a fair coin that gave us heads 525 or more times.

$$\begin{aligned}
\mathbb{P}(S_n > 525|p = 0.5) &\overset{*}{=} \mathbb{P}(S_n > 524.5|p = 0.5) \\
&= \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} > \frac{524.5 - 1000(0.5)}{\sqrt{1000(0.5)(0.5)}}\right) \\
&= \mathbb{P}(Z > 1.549516) \\
&= 1 - \mathbb{P}(Z < 1.549516) \\
&= 1 - 1.549516 = 0.0606289
\end{aligned}$$

Hence the probability is 6.06289%.

Remark *: we used continuity correction. We use this for part b) too.

```
> 1 - pnorm((524.5 - 500)/ sqrt(1000 * 0.5 * 0.5))
[1] 0.06062886
```

Solution Part 2

(b) I assume the author is asking for the probability of getting a false conclusion when the coin is actually biased. This means that we would conclude we have a fair coin when we don't, so we obtain less than 525 heads.

$$\begin{aligned}
\mathbb{P}(S_n \leq 524|p = 0.55) &\overset{*}{=} \mathbb{P}(S_n < 524.5|p = 0.55) \\
&= \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} < \frac{524.5 - 1000(0.55)}{\sqrt{1000(0.55)(0.45)}}\right) \\
&= \mathbb{P}(Z < -1.620886) \\
&= 0.05252104
\end{aligned}$$

Hence the probability is 5.252104%.

```
> pnorm((524.5 - 1000 * 0.55)/ sqrt(1000 * 0.55 * 0.45))
[1] 0.052521
```

Question 5 (Section 5 Question 28)

Twelve percent of the population is left handed. Approximate the probability that there are at least 20 left-handers in a school of 200 students. State your assumptions.

Solution

Here we will use the binomial approximation to a normal distribution, i.e., using the property that if S_n denotes the number of successes then:

$$\frac{S_n - np}{\sqrt{np(1-p)}} \sim N(0,1)$$

Where here S_n represents the number of left handed people. We have $n(p)(1-p) = 200 * 0.12 * 0.88 = 21.12 \geq 10$. Hence the assumption is satisfied and we can use the binomial approximation to normal. Hence,

$$\begin{aligned}
\mathbb{P}(S_n \geq 20) &\overset{*}{=} \mathbb{P}(S_n > 19.5) \\
&= \mathbb{P}\left(\frac{S_n - np}{\sqrt{np(1-p)}} > \frac{19.5 - 200(0.12)}{\sqrt{200 * 0.12 * 0.88}}\right) \\
&= \mathbb{P}(Z > -0.9791868) \\
&= 1 - \mathbb{P}(Z < -0.9791868) \\
&= 1 - 0.1637438 = 0.8362562
\end{aligned}$$

So the probability that there are at least 20 left handers is 83.62562%. Again, (*) represents the part where we use the contiunity correction.

Remark: the $n(p)(1-p) \geq 10$ assumption is actually subjective; some textbooks may suggest a different threshold, such as $n(p) \geq 5, n(1-p) \geq 5$. However, the lecture mentions the same assumption as the textbook. Hence, you must be careful with content online.

```
> 1 - pnorm((19.5 - 200*(0.12))/(sqrt(200 * 0.12 * 0.88)))
[1] 0.8362562
```